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REPRESENTATION OF A LATTICE BY A GRAPH WITH RESPECT TO AN IDEAL AND ITS CHARACTERIZATION

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Abstract. In this paper a graph of a finite lattice L with respect to an ideal I is defined and is denoted by $G_I(L)$. We demonstrate essential findings that connect ideals and graph representations. Furthermore, we explore the construction of the incidence matrix P for $G_I(L)$ and utilize it as a context table to derive the corresponding Concept Lattice L_P. In doing so, we uncover intriguing relationships between $G_I(L)$ and L_P.

The results obtained in this study offer valuable insights into the interactions between finite lattices, their graph representations, and the significance of prime ideals. The implications of these findings can have applications in various domains, such as lattice theory and graph theory.

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1. Introduction

Making connection between various algebraic structures and graph theory by assigning graphs to an algebraic structure and investigating the properties of one from the another is an exciting research methods in recent days. A. Berry et al. represented the concept lattice in terms of graph and established the relations and explained how to use binary relations to generate graphs [1]. In the same article, authors presented a new relationship between lattices and graphs: given a binary relation R, they defined an underlying graph G_R, and established a one-to-one correspondence between the set of elements of the concept lattice of R and the set of minimal separators of G_R. The article also explained how to use the properties of minimal separators to define a sublattice, decompose a binary relation,and generate the elements of the lattice. Further, F. Hao et al. explained real-world applications information represented in terms of graphs for the better analysis [3]. In addition to this, Formal Concept Analysis (FCA), a mathematical theory oriented at applications in knowledge representation, knowledge acquisition, data analysis and visualization. It provides tools for understanding the data by representing it as a hierarchy of concepts or more exactly, a concept lattice. FCA can help in processing a wide class of data types providing a framework in which various data analysis and knowledge acquisition techniques can be formulated. There are many approaches developed to give relation connecting graph with concept lattice. S. Bhavanari et al. defined and studied graph of a nearring N with respect to an ideal I of N, denoted by $G_I(N)$. Further they defined a new type of symmetry called ideal symmetry of $G_I(N)$ and proved that ideal symmetry of $G_I(N)$ implies by the symmetry determined by the automorphism group of $G_I(N)$ [6]. In this paper we discuss all these results with respect to a finite lattice.

In this article, Section 2 provides basic notations and definitions. We introduced a concept called the graph of a lattice L with respect to an ideal I, denoted by $G_I(L)$ and prime graph of a lattice. We prove that the prime graph of a lattice L is a subgraph of L with respect to any ideal of L in Section 3.
In Section 4, relation between $G_I(L)$ and concept lattice L_P were discussed. We characterized star graph with respect to concept lattice L_P.We prove that, if L is a lattice with a unique atom, then the graph of a lattice with respect to the ideal I={0} is always a star graph. Few more results are also discussed connecting concept lattice and $G_I(L)$.

2. Notations and Definitions

For the following notations and definitions, reader may refer the book [4], [5] and [2].
Throughout the paper G represents finite graph. The *vertex connectivity* of G is defined by $k(G) = \min \{n \geq 0 :$ there exisits a vertex cut $S \subseteq V(G)$ such that $|S| = n\}$ if G has a finite vertex cut and $k(G) = \infty$ otherwise. Similarly, the *edge connectivity* of G is defined by $\lambda(G) = \min \{n \geq 0 :$ there exisits an edge cut $T \subseteq E(G)$ such that $|S| = n\}$ if G has a finite edge and $\lambda(G) = \infty$ otherwise. Let $\delta(G) = \min \{deg(v) : v \in V\}$, then the following result always holds; $k(G) \leq \lambda(G) \leq \delta(G)$.
In this sequel, L denotes a finite lattice with minimum element 0 and maximum element 1. An element a of a lattice L is called an atom of L, if L has the minimum element 0 and 0 is covered by a. A lattice L is said to be atomistic if every non zero element of L is join of atoms contained in it.
An ideal I of a lattice L is a non-empty subset of L satisfying the conditions;

i) $a, b \in I$ implies $a \vee b \in I$.

ii) $a \in I$ and $x \in L$ implies $a \wedge x \in I$.

An Ideal I of a lattice L is called prime if x, $y \in L$, $x \wedge y \in I$ implies that $x \in I$ or $y \in I$.
A prime ideal I of of a lattice L is said to be a minimal prime ideal if there is no prime ideal which is properly contained in I.
A formal context or simply a context $T = (G, M, I)$ consists of two sets G and M and a relation I between G and M. The elements of G are called objects of T, the elements of M are called attributes of T. If an objects a has attribute m, we denote it by aIm. A 1 in row a and column m means that the object a has attribute m. A context is represented in terms of binary matrix.

For a set $A \subseteq G$ of objects and $B \subseteq M$ of attributes, $A' = \{m \in M : aIm$ for all $a \in A\}$, $B' = \{g \in G : bIg$ for all $b \in B\}$. A formal concept of the context $T = (G, M, I)$ is a pair (A, B) with $A \subseteq G$, $B \subseteq M$, $A' = B$ and $B' = A$. $L(G, M, I)$ denotes the set of all concepts of the context T. The set of all concepts, when ordered by set-inclusion, satisfies the properties of a complete lattice. The lattice of all concepts is called concept lattice. A context table T is presented in Figure 1 and corresponding formal concept lattice is shown in Figure 2.

A context (G, M, I) is called clarified, if for any $g, h \in G$ from $g^{'} = h^{'}$, it always follows that $g = h$ and correspondingly, $m^{'} = n^{'}$ implies $m = n$ for all $m, n \in M$.
A clarified context (G, M, I) is called row reduced, if every object concept is $\vee - irreducible$

	1	**2**	**3**	**4**
a	1	0	0	1
b	1	1	0	0
c	1	0	1	0
d	0	1	0	1

Figure 1. Context T

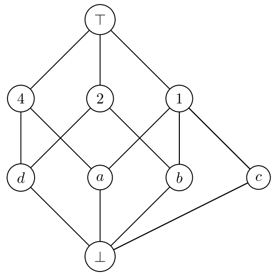

Figure 2. Concept lattice

and column reduced, if every attribute concept is $\wedge - irreducible$. A context, which is both row reduced and column reduced is called reduced context.

3. Graph of a lattice with respect to an Ideal

In this section, we introduce a concept called *Graph of a lattice with respect to an Ideal* and this notion in line with the notion defined by S. Bhavanari et al. for nearring in [6].

Definition 3.1. Let I be an ideal of L. We define, the graph of L with respect to I is a graph with each element of L as a vertex, and two distinct vertices x and y are connected by an edge if and only if $x \wedge y \in I$. We denote the graph of L with respect to I by $G_I(L)$. In particular the prime graph of a lattice L is a graph with respect to the ideal $I = \{0\}$.

Let us consider a lattice L in Figure 3, having the ideals I_1=$\{0\}$, $I_2 = \{0, b\}$, $I_3 = \{0, a\}$, $I_4 = L$. Figure 4 to Figure 7 are the graphs of a lattice with respect to the ideals I_1 to I_4 respectively. It is clear that except ideal I_1, all other ideals are prime.

Remark 3.2. The following consequences are easy to verify:

(a) $G_I(L)$ is a connected graph without self-loops and multiple edges.
(b) The maximum distances between any two vertices of $G_I(L)$ is at most two.

If an Ideal $I \subseteq J$, then the graphs $G_I(L)$, $G_J(L)$ are related with the following relation. Similar proposition for nearring is derived in [6] by S. Bhavanari et al.

Proposition 3.3. *Let I and J be ideals of L such that $I \subseteq J$. Then, $G_{\{0\}}(L) \sqsubseteq G_I(L) \sqsubseteq G_J(L) \sqsubseteq G_L(L)$.*

Definition 3.4. Let G be a graph with respect to vertex set $V(G)$. Then, *strong vertex cut* of a graph G is a subset $S \subseteq V(G)$ such that the graph $G - S$ is totally disconnected. The *strong vertex connectivity* of G is defined as, $K(G) = \min\{n \geq 0 :$ there exisits a strong vertex cut $S \subseteq V(G)$ such that $|S| = n\}$.

Remark 3.5.
(a) It is easy to verify that, $k(G) \leq K(G)$.
(b) Consider the graph $G_{I_1}(L)$ given in Figure 4. We can observe that $\{0\}$ is a vertex cut but not a strong vertex cut of $G_{I_1}(L)$. Hence $k(G_{I_1}(L)) = 1$ and $K(G_{I_1}(L)) = 2$.

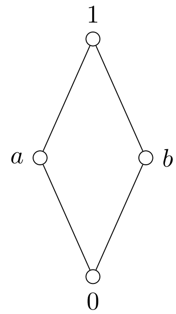

Figure 3. Lattice L

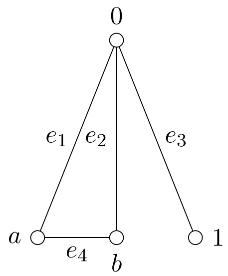

Figure 4. $G_{I_1}(L)$

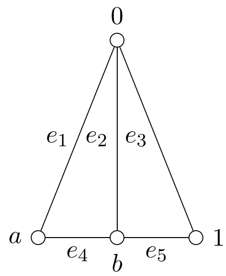

Figure 5. $G_{I_2}(L)$

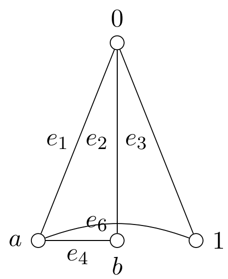

Figure 6. $G_{I_3}(L)$

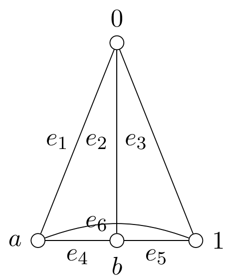

Figure 7. $G_{I_4}(L)$

Theorem 3.6. *Let I be an ideal of L. I is prime if and only if I is a strong vertex cut of $G_I(L)$.*

Proof. Let I be an prime ideal of L. If $I = L$, then there is nothing to prove. Let $I \neq L$ and $x, y \in L \setminus I$ such that $x \neq y$. If possible suppose that there exists an edge between x and y in $G_I(L)$. Then $x \wedge y \in I$. As I is prime ideal of L either x or $y \in I$. This is a contradiction since $x, y \notin I$.

Conversely, suppose that I is a strong vertex cut of $G_I(L)$. To prove I is prime, take $x , y \in L$ such that $x \wedge y \in I$. Let, $x \neq y$. If possible, suppose that $x \in L \setminus I$ and $y \in L \setminus I$. As I is a strong vertex cut of $G_I(L)$, there is no edge between x and y in $G_I(L)$. This implies $x \wedge y \notin I$, a contradiction. Thus I is a prime ideal of L.

□

Corollary 3.7. *Let I be a minimal prime ideal of L. Then, $K(G_I(L)) = |I|$.*

Lemma 3.8. *Let I be a prime ideal of a lattice L. Then $x \in I$ if and only if deg(x)=deg(0) in $G_I(L)$.*

Proof. Suppose that deg(x)=deg(0). Then $x \wedge y \in I$ for all $y \in L$ such that $y \neq x$. If $I = L$ then $x \in I$. Let $I \neq L$. Choose $y \in L \setminus I$. As I is a prime ideal and $x \wedge y \in I$, we get $x \in I$. Conversely, let $x \in I$. If $x = 0$, then the result is true. Let $x \neq 0$. If possible suppose that deg(x) < deg(0). Then there exists a vertex y such that y is not adjacent to x in $G_I(L)$. This implies $x \wedge y \notin I$. Now, $x \wedge y \leq x \in I$, and I is an ideal implies $x \wedge y \in I$, a contradiction. This proves deg(x)=deg(0).

□

Proposition 3.9. *Let I be an ideal of L such that $|L| = n$ and $|I| = m$.*

(1) *If I is proper ideal of L then $k(G_I(L)) = \lambda(G_I(L)) = \delta(G_I(L)) = m$.*
(2) *If $I = L$ then $k(G_I(L)) = \lambda(G_I(L)) = \delta(G_I(L)) = n - 1$.*

Proof. For any graph G, it is well known that $k(G_I(L)) \leq \lambda(G_I(L)) \leq \delta(G_I(L))$. To prove (1), suppose that I is a proper ideal of L. Clearly, the maximum element 1 of the lattice L does not belongs to I. In $G_I(L)$, 1 is the vertex having minimum degree and 1 is connected to all vertices of ideal I. Therefore deg(1)=$\delta(G_I(L)) = m$. Also if we remove all vertices of I the graph becomes disconnected, therefore $k(G_I(L)) = m$, proving (1).

If $I = L$. Then $G_I(L)$ is a complete graph. Clearly $k(G_I(L)) = \lambda(G_I(L)) = \delta(G_I(L)) = n-1$, proving (2).

□

The following notion is in line with the notion of ideal symmetric defined for nearring by S Bhavanari et al. in [6].

Definition 3.10. The graph $G_I(L)$ is said to be *ideal symmetric* if for every $\overline{xy}$ in $G_I(L)$ either deg(x)=deg(0) or deg(y)=deg(0).

Remark 3.11. The graph in Figure 4 is not ideal symmetric, whereas other graphs (Figure 5 to Figure 7) are ideal symmetric.

Theorem 3.12. *An ideal I of a lattice L is prime if and only if $G_I(L)$ is ideal symmetric.*

Proof. Let I be a prime ideal of L. Let $\overline{xy} \in G_I(L)$. Then $x \wedge y \in I$. As I is prime either $x \in I$ or $y \in I$. Now by Lemma 3.8 we get deg(x)=deg(0) or deg(y)=deg(0). Thus $G_I(L)$ is ideal symmetric. Conversely, suppose that $G_I(L)$ is ideal symmetric. To prove that I is prime, let $x, y \in L$ and $x \wedge y \in I$. Now there exists an edge between x and y in $G_I(L)$. As $G_I(L)$ is ideal symmetric, deg(x)=deg(0) or deg(y)=deg(0). This implies $x \in I$ or $y \in I$. Thus I is prime ideal of L.

□

4. Relation between $G_I(L)$ and the concept lattice L_P

Throughout this section, L denotes a finite lattice with n elements. Let P be the incidence matrix of $G_I(L)$ and L_P be its concept lattice.

Now, consider the graph $G_{I_1}(L)$ in Figure 4. The incidence matrix P of $G_{I_1}(L)$ is in Figure 8 and its concept lattice L_P is given by Figure 9.

$$
\mathrm{P} = \begin{array}{c} \\ 0 \\ a \\ b \\ 1 \end{array}
\begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\
\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}
$$

Figure 8. Incidence matrix P of $G_{I_1}(L)$

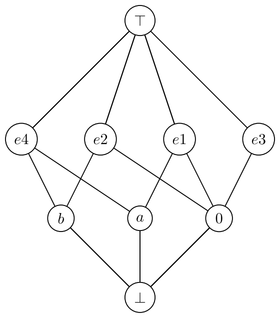

Figure 9. Concept lattice L_P

While observing the transformation of graph to a lattice, we found some relations which will be discussed in the following theorems.

Theorem 4.1. *Let P denotes incidence matrix of a star graph G. Then the corresponding concept lattice L_P is always a unique atom lattice. Conversely, if L is a unique atom lattice then $G_{\{0\}}(L)$ is a Star graph.*

Proof. Let G be a star graph with n vertices, P denotes its incidence matrix. Let x be the center of star graph (which is connected by all other vertices by a unique edge). We shall prove that x is the unique atom of L_P. Clearly, x is a join- irreducible element of L_P. If y is a row of P, then $y' \subseteq x'$, hence the element $y \geq x$ in L_P. Which implies all other join-irreducible elements of L_P are greater than or equal to x, proving that x is the only atom of L_P.
On the other hand, let a be unique atom of L and $G_{\{0\}}(L)$ be the graph of the lattice with respect to the ideal $'0'$. We shall prove that $G_{\{0\}}(L)$ is a Star graph. Note that $'0'$ is connected to all vertices of $G_{\{0\}}(L)$.
Suppose that, $x \neq 0$, $y \neq 0$, then $x \wedge y \geq a \neq 0$. This implies there is no edge between x and y in $G_{\{0\}}(L)$. Hence $G_{\{0\}}(L)$ is a Star graph. □

The following corollary states isomorphic relation between graph of two lattices with respect to their ideals.

Corollary 4.2. *If L_1 and L_2 are two lattices with unique atom and $|L_1| = |L_2|$, then $G_{\{0\}}(L_1) \backsimeq G_{\{0\}}(L_2)$.*

Proof. Let, L_1 and L_2 are two lattices with unique atom and $|L_1| = |L_2|$, then by the *theorem 4.1*, $G_{\{0\}}(L_1)$ and $G_{\{0\}}(L_2)$ are star graphs, which preserves same structure.
Therefore, $G_{\{0\}}(L_1) \backsimeq G_{\{0\}}(L_2)$. □

Now, in the following theorem we characterize the complete graph with respect to $G_I(L)$ for any ideal I.

Theorem 4.3. *Let L is not a co atom lattice. $G_I(L)$ is a complete graph if and only if $I = L$.*

Proof. Let $G_I(L)$ is a complete graph. We need to prove that the corresponding ideal I must be equal to lattice L. Now, for any $a, b \in G_I(L)$, a and b are connected by an edge in $G_I(L)$ iff $a \wedge b \in I$, in particular $a \wedge 1 = a \in I$. Therefore, except the maximal element 1 all other elements are in I. Since L is not a unique dual atom lattice, there exists atleast two elements a , $b \in L$ which are dual atoms ($a \neq 1$, $b \neq 1$) and $a \in I$, $b \in I$. This implies $a \vee b = 1 \in I$. Proving that $I = L$.
On the other hand, if $I = L$, then all the vertices of $G_I(L)$ are connected by a unique edge.Therefore, $G_I(L)$ is complete graph. □

Theorem 4.4. *Let, G be an simple graph and P be its corresponding incidence matrix. Then the associated concept lattice L_P is always an dual atomistic lattice.*

Proof. Let G be an simple graph. In the incidence matrix P, since each edge e_i is connected by atleast one different vertex, each column is independent. In a clarified, reduced context if each row is independent then the corresponding lattice is always an atomistic lattice. Therefore, by the duality principle it follows that the corresponding concept lattice L_P of the graph G is a dual atomistic lattice. □

5. Conclusion

In this paper we introduced the graph of a lattice with respect to an ideal and obtained interesting results. Moreover, an attempt has been made to connect the graph theory, the lattice theory and formal concept analysis.

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